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The class of universality of integrable and isotropic $GL(N)$ mixed magnets

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Abstract. We discuss a class of transfer matrix built by a particular combination of isomorphic and non-isomorphic $GL(N)$ invariant vertex operators. We construct a conformally invariant magnet constituted of an alternating mixture of $GL(N)$ 'spins' operators at different orders of representation. The corresponding central charge is calculated by analysing the low-temperature behaviour of the associated free energy. We also comment on possible extensions of our results for more general classes of mixed systems.

1. Introduction

One of the most useful methods of constructing an integrable one-dimensional quantum spin chain has been to find solutions of the Yang–Baxter [1, 2] equations. In the context of the lattice models, such solutions define vertex operators acting on the tensor product of two vector spaces $V \otimes h_\alpha$ of the local states on the horizontal and vertical lines of a given site α of the two-dimensional lattice. The complete Hilbert space of the model on a lattice of L sites is $\prod_{\alpha=1}^{L} \otimes h_\alpha$. The vector space V is an auxiliary space which is useful in the definition of the associated transfer matrix. Defining $R^V_{h,\alpha}(\mu)$ as such vertex operator, the corresponding transfer matrix can be expressed by [2, 3]

$$T(\mu) = \mathrm{Tr}_V[R^V_{h,L}(\mu) R^V_{h,L-1}(\mu) \cdots R^V_{h,1}(\mu)] \tag{1}$$

where the trace is carried out in the auxiliary space V and μ is a variable which parametrizes the Yang–Baxter solution.

A class of important solutions of the Yang–Baxter equation is isomorphic on the horizontal and vertical spaces ($V \equiv h_\alpha$) and rational on the spectral parameter μ. One example is the vertex operator composed of generators which are invariant by some of the semi-simple A, D, E Lie algebra at a certain order k of its representation. The main feature of these solutions is that they are believed to define conformally invariant quantum spin chains [4–6] which realize the class of universality of Wess–Zumino–Witten–Novikov (WZWN) [7] field theories with topological charge $\tilde{k}$. Recently, de Vega and Woynarovich [8] have pointed out that other interesting classes of conformally invariant models can be constructed. These theories are obtained by combining isomorphic and non-isomorphic vertex operators invariant by some group symmetry at different orders of representation. As a typical example one can consider an alternating combination of an isomorphic vertex at even sites and non-isomorphic operators at odd sites. Considering that the isomorphic (non-isomorphic) vertex operator acts on the vector space $V^{(k)} \otimes h^{(k)}_\alpha$ ($V^{(k)} \otimes h^{(k')}_\alpha$) of order

k and k', the theory described above will lead to a mixed magnet chain with 'spin' operators of order k and k'. Following this approach, de Vega and Woynorovich [8] have constructed an alternating anisotropic Heisenberg chain of spins 1/2 and 1 and analysed its behaviour in the thermodynamic limit. More recently, the central charge of the conformally invariant isotropic $SU(2)$ mixed chain has been computed independently in [9, 10] for spins 1/2–1 and 1/2–S, respectively.

Its seems interesting to investigate the critical properties of conformally invariant mixed spin chain for a more general class of group symmetry and its possible relations with WZWN field theories. In this paper we study an isotropic alternating $GL(N)$ model in which the isomorphic vertex is in the fundamental representation ($k = 1$) and the non-isomorphic operator is defined in the tensor space product of the fundamental and the order k of representations. By using the thermodynamic Bethe ansatz, we have calculated the associated central charge in the case of a conformally invariant mixed $GL(N)$ system. It is noted that this central charge can be decomposed in terms of the conformal anomaly of two WZWN theories with different topological charges $\tilde{k} = 1$ and $\tilde{k} = k - 1$, respectively. We also discuss the generalization of our results in the case of semi-simple Lie algebras at arbitrary symmetric representation.

This paper is organized as follows. In section 2 we define the $GL(N)$ mixed magnet and we discuss its diagonalization by the quantum inverse scattering (QIS) formalism. In section 3 we use the thermodynamic Bethe ansatz (TBA) approach in order to investigate the low-temperature behaviour of a conformally invariant mixed $GL(N)$ system. Section 4 is devoted to our discussion on possible extensions of the results of section 3 to other Lie algebras. Section 5 contains our conclusions. In appendixes A and B we summarize some details of the QIS approach and we present extra numerical checks of the finite-size behaviour for the ground-state energy, respectively.

2. The mixed $GL(N)$ integrable model

A rational $GL(N)$ invariant vertex operator has been found by Kulish *et al* [11] in their study of group invariant solutions of the Yang–Baxter relation. The $GL(N)$ non-isomorphic vertex defined on the fundamental and on the symmetric order k of representation is given by the expression [11]

$$R^{1}_{k,\alpha}(\mu) = \mu - (k-1)/2 + \sum_{i,j} e^{ij} \otimes E^{ji}_{\alpha} \tag{2}$$

where $e^{ij}(E^{ij})$ are the generators of $GL(N)$ in the fundamental (order k) representation. For the fundamental representation the matrix elements of e^{ij} are $(e^{ij})_{kl} = \delta_{il}\delta_{jk}$. We also notice the identity $R^{1}_{1,\alpha}(0) = \mathcal{P}$, where $\mathcal{P}$ is the permutation operator $\mathcal{P}V \otimes h_{\alpha} = h_{\alpha} \otimes V$.

The $GL(N)$ mixed system is defined in terms of its transfer matrix of alternating isomorphic ($k = 1$) and non-isomorphic vertex operators by

$$T_{1,k}(\mu) = \mathrm{Tr}_{V^{(1)}}[\tau_{1,k}(\mu)] \qquad \tau_{1,k}(\mu) = R^{1}_{1,L}(\mu) R^{1}_{k,L-1}(\mu) \cdots R^{1}_{1,2}(\mu) R^{1}_{k,1}(\mu) \tag{3}$$

where the matrix product and the trace are defined in the auxiliary space $V^{(1)}$ of the elements e^{ij} and $\tau_{1,k}(\mu)$ is the so-called monodromy matrix. The associated one-dimensional quantum Hamiltonian acting on the Hilbert space $\prod_{\alpha=1}^{L} h_{\alpha}$ is determined by the logarithm derivative of the transfer matrix, $H_{1,k} = J\mathrm{d}[\ln(T_{1,k}(\mu))]|_{\mu=0}/\mathrm{d}\mu$. By using this relation and some properties of the $GL(N)$ group we find

$$H_{1,k} = \frac{\tilde{J}}{2}\sum_{n=\mathrm{odd}}^{L-1}\Bigg[\sum_{i,j,l,m}^{N} e_{n-1}^{ij}\{E_n^{jm}, E_n^{li}\}e_{n+1}^{ml} - (k-1)\sum_{i,j,l}^{N}(e_{n-1}^{ij}E_n^{jl}e_{n+1}^{li} + e_{n-1}^{il}E_n^{ji}e_{n+1}^{lj})$$
$$+ \frac{(k-1)^2}{2}\sum_{i,j}^{N} e_{n-1}^{ij}e_{n+1}^{ji} + \sum_{i,j}^{N}(e_{n-1}^{ij}E_n^{ji} + E_n^{ij}e_{n+1}^{ji}) - (k-1)\Bigg]$$
$$- \frac{JL}{2}\left(1 + \frac{2}{k+1}\right) \tag{4}$$

where $\tilde{J} = 4J/(k+1)^2$ and we have conveniently added an extra constant $-JL(1 + 2/k+1)/2$. In this paper we are interested in the antiferromagnetic properties of this model, and therefore we have chosen $J = 1$. Setting $k = 1$ in equation (4) we reproduce the results of [12, 13] and for $N = 2$ and $k = 2S$ we recover previous calculations for the mixed Heisenberg chain [8–10].

The diagonalization of the transfer matrix $T_{1,k}(\mu)$ (or the one-dimensional Hamiltonian $H_{1,k}$) follows from the generalization of the QIS approach [3] applied to multi-state vertex models [13–15]. In the QIS construction the definition of the pseudo vacuum and the block form of the monodromy matrix are two important ingredients of this method. In our case the pseudo vacuum $|0\rangle$ is defined by

$$|0\rangle = |0\rangle_1^k \otimes |0\rangle_2^1 \otimes \cdots \otimes |0\rangle_{L-1}^k \otimes |0\rangle_L^1 \tag{5}$$

where $|0\rangle_\alpha^k$ is the vector of highest weight of the $GL(N)$ algebra at order k of the symmetric representation acting on the site α, namely $E_\alpha^{ij}|0\rangle_\alpha^k = k\delta_{i,1}\delta_{j,1}|0\rangle_\alpha^k$. An important property of this state is that $\tau_{1,k}(\mu)|0\rangle$ has the following block triangular form

$$\tau_{1,k}(\mu)|0\rangle = \begin{pmatrix} (\mu+1)^{L/2}(\mu+(k+1)/2)^{L/2} & B^i(\mu) \\ 0 & \delta_{i,j}(\mu)^{L/2}(\mu-(k-1)/2)^{L/2} \end{pmatrix}|0\rangle \tag{6}$$

where $B^i(\mu) \equiv \tau_{1,k}(\mu)^{1,i+1}$, $i = 1, \ldots, N-1$.

Following the QIS machinery [13–15] a certain linear combination of the states $B^{j_1}(\mu_1^1)\cdots B^{j_n}(\mu_n^1)|0\rangle$ can be considered as a basis for the eigenstates of the transfer matrix, provided that the parameters $\{\mu_1^1, \cdots, \mu_n^1\}$ satisfy a set of non-linear equations denominated Bethe ansatz equations. The technical steps are fairly parallel with those of [13–15] and in appendix A we have collected some of the details. Defining the convenient shift $\mu_j^r = \mathrm{i}\lambda_j^r - r/2$, the Bethe ansatz equations are given by

$$\left(\frac{\lambda_j^1 - \mathrm{i}/2}{\lambda_j^1 + \mathrm{i}/2}\right)^{L/2}\left(\frac{\lambda_j^1 - \mathrm{i}k/2}{\lambda_j^1 + \mathrm{i}k/2}\right)^{L/2}$$
$$= -\prod_{l=1}^{M^1}\frac{\lambda_j^1 - \lambda_l^1 - \mathrm{i}}{\lambda_j^1 - \lambda_l^1 + \mathrm{i}}\prod_{l=1}^{M^2}\frac{\lambda_j^1 - \lambda_l^2 + \mathrm{i}/2}{\lambda_j^1 - \lambda_l^2 - \mathrm{i}/2}\prod_{l=1}^{M^r}\frac{\lambda_j^r - \lambda_l^r - \mathrm{i}}{\lambda_j^r - \lambda_l^r + \mathrm{i}}$$
$$= \prod_{l=1}^{M^{r+1}}\frac{\lambda_j^r - \lambda_l^{r+1} - \mathrm{i}/2}{\lambda_j^r - \lambda_l^{r+1} + \mathrm{i}/2}\prod_{l=1}^{M^{r-1}}\frac{\lambda_j^r - \lambda_l^{r-1} - \mathrm{i}/2}{\lambda_j^r - \lambda_l^{r-1} + \mathrm{i}/2} \tag{7}$$

where $r = 2, \ldots, N-1$ and we define $M^N \equiv 0$. The eigenvalues of the Hamiltonian $H_{1,k}$ are parametrized by

$$E_{1,k} = -\sum_{j=1}^{M^1} \frac{1}{(\lambda_j^1)^2 + (1/2)^2}. \tag{8}$$

In principle, the construction discussed above can be carried out for the 'dual' one-dimensional $H_{k,1}$ spin chain. The main task is to find an isomorphic $GL(N)$ invariant operator $R_{k,\alpha}^k(\mu)$ in the symmetric representation of order k. Using the approach of [11], Johannesson [16] has explicitly exhibited such operator. Hence, analogously to equation (3), the associated transfer matrix $T_{k,1}(\mu)$ is expressed by

$$T_{k,1}(\mu) = \mathrm{Tr}_{V^{(k)}}[\tau_{k,1}(\mu)] \qquad \tau_{k,1}(\mu) = R_{1,L}^k(\mu) R_{k,L-1}^k(\mu) \cdots R_{1,2}^k(\mu) R_{k,1}^k(\mu) \tag{9}$$

where $V^{(k)}$ is the space of the matrices E^{ij} of $GL(N)$ at order k of representation.

An important property of $T_{1,k}(\mu)$ and $T_{k,1}(\mu')$ is their commutativity for arbitrary values of the parameters μ and μ'. This property follows from the 'mixed' Yang–Baxter relation satisfied by the non-local vertices† $R_k^k(\mu)$ and $R_k^1(\mu)$. As a consequence, the Hamiltonians $H_{1,k}$ and $H_{k,1}$ can be simultaneously diagonalized and their eigenspectrum are parametrized by the same Bethe equations, i.e. equations (7). However, the eigenenergies of $H_{k,1}$ [16] are expressed in terms of the variables $\{\lambda_j^1\}$ by a different function of that of equation (8), namely

$$E_{k,1} = -\sum_{j=1}^{M^1} \frac{k}{(\lambda_j^1)^2 + (k/2)^2}. \tag{10}$$

At this point, it is important to remark that the transfer matrix $T_{1,k}(\mu)$ and its 'dual' $T_{k,1}(\mu)$ are not rotational invariant in the horizontal/vertical space of states. A simple way of defining [8] a 'mixed' symmetric transfer matrix $T^{\mathrm{sym}}(\mu)$, preserving the rotational invariance, is by formally multiplying these two transfer matrices

$$T^{\mathrm{sym}}(\mu) = T_{1,k}(\mu) T_{k,1}(\mu). \tag{11}$$

Due to the commutativity between $T_{1,k}(\mu)$ and $T_{k,1}(\mu)$ the spectrum of $T^{\mathrm{sym}}(\mu)$ is clearly parametrized by the same Bethe ansatz equation and the eigenvalues of the corresponding one-dimensional Hamiltonian H^{sym} are added,

$$E^{\mathrm{sym}} = -\sum_{j=1}^{M^1} \frac{1}{(\lambda_j^1)^2 + (1/2)^2} - \sum_{j=1}^{M^1} \frac{k}{(\lambda_j^1)^2 + (k/2)^2}. \tag{12}$$

In this sense equations (7), (8), (10) and (12) define three families of possible spectrums parametrized by a single Bethe ansatz equation. For instance, in the case of small size L, one only needs to solve equation (7) and compare it with the exact solution of the spectrum of $H_{1,k}$ in order to investigate the structure of the variables $\{\lambda_j^r\}$ which are going to parametrize the whole spectrum of all these three families of models. In general, we remark that a certain solution $\{\lambda_j^r\}$ of equation (7) will not necessarily produce the same ith

† The vertex $R_k^{k'}(\mu)$ acts in the non-local space $V^{(k')} \otimes h$.

state ordered in energy for all these models. The ground state, however, is characterized by the same structure of $\{\lambda_j^r\}$ for all these three families. In the thermodynamic limit, $L \to \infty$, the ground-state solution $\{\lambda_j^r\}$ is composed by a mixture of real numbers η_j^r and a set of complex roots $\lambda_{j,\alpha}^r$. The complex structure $\lambda_{j,\alpha}^r$ cluster in the so-called k-string form

$$\lambda_{j,\alpha}^r = \xi_j^r + \tfrac{1}{2}\mathrm{i}(k+1-2\alpha) \qquad \alpha = 1, 2, \cdots, k \tag{13}$$

where ξ_j^r is a real parameter denoting the centre of the k-string.

In order to calculate the ground-state energy in the $L \to \infty$ limit, one has to solve the Bethe ansatz equations (7) for the variables η_j^r and ξ_j^r. Taking into account equation (13) and after some standard manipulations, we are able to compute the following values for the ground-state energy per particle

$$e_\infty^{1,k} = -\frac{1}{N}\left[\psi(1) - \psi(1/N) + \psi\left(\frac{k-1+2N}{2N}\right) - \psi\left(\frac{k+1}{2N}\right)\right] \tag{14}$$

$$e_\infty^{k,1} = -\frac{1}{N}\left[\psi(1) - \psi(k/N) + \psi\left(\frac{k-1+2N}{2N}\right) - \psi\left(\frac{k+1}{2N}\right) + \sum_{j=1}^{k-1}\frac{N}{j}\right] \tag{15}$$

where $\psi(x)$ is the Euler psi function. From equation (12) it is clear that $e_\infty^{\mathrm{sym}} = e_\infty^{1,k} + e_\infty^{k,1}$.

Let us now concentrate our attention on the low-lying excitations of the symmetric mixed model. Since the transfer matrix $T^{\mathrm{sym}}(\mu)$ is rotational invariant, its associated quantum spin Hamiltonian is a strong candidate to be conformally invariant. We recall that the analysis of the critical properties in spin chains depends on the behaviour of its dispersion relation for low momenta p. The computation of the dispersion relation follows the standard formalism of perturbing the ground-state structure by holes and string of arbitrary length (see e.g. [12]). The only subtle fact is that in this alternating mixed system the total momentum is half of that considered in homogeneous models ($k = 1$) [10]. Following [12, 16, 8] we find that the dispersion relation for all branches of excitations [12, 16] possess the same linear behaviour for the total low momenta p

$$\varepsilon(p) = \frac{4\pi}{N}p. \tag{16}$$

From equation (16) the sound velocity is $v_s = 4\pi/N$, independent of the order k of representation. We observe that v_s is double of that appearing in the homogeneous model ($k = 1$) [12, 16]. This fact can be easily interpreted by noting that the elementary translation 'cell' of the alternating mixed models is double that of a homogeneous system. Indeed, considering this discussion and the previous results of [12, 16] (assuming independence of k) leads us to guess that $v_s = 4\pi/N$, with no need of an explicit computation†.

In the next section we are going to compute the conformal anomaly which defines the class of universality of these conformal mixed $GL(N)$ models.

† In fact, for an alternating model with periodicity l (made by a collection of 'spins' operators at different order of representation), we should have $v_s = 2l\pi/N$.

3. The thermodynamics of the mixed $GL(N)$ model

In order to discuss the thermodynamic properties we adopt the thermodynamic Bethe ansatz approach originally proposed by Yang and Yang [17]. This method is based on the minimization of the free energy and takes advantage of the integrability through the Bethe ansatz equations. The first step is to notice that the Bethe ansatz equations (7) admit the same string hypothesis used previously by Takahashi [18] in the isotropic Heisenberg model. This observation allows us to conclude that, in the $L \to \infty$ limit, the parameters are λ_j^r organized in strings of the type described in equation (13). Substituting equation (13) into equation (7) and taking the thermodynamic limit, we are able to obtain the following infinity set of coupled integral equations for the densities $\sigma_n^r(\lambda)$ ($\tilde{\sigma}_n^r(\lambda)$)of particles (holes)

$$\tilde{\sigma}_n^r(\lambda) = \frac{\delta_{r,1}}{4\pi}[\phi_{n,1/2}(\lambda) + \phi_{n,k/2}(\lambda)] - \sum_{r'=1}^{N-1}\sum_{j=1}^{\infty}(A_{n,j} * B_{r,r'} * \sigma_j^{r'})(\lambda) \tag{17}$$

where n indicates the length of the n-string, and $(f * g)(x)$ denotes the convolution $(1/2\pi)\int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y$. The functions $A_{n,j}(\lambda)$, $B_{r,r'}(\lambda)$ and $\phi_{n,j}(\lambda)$ are easily represented in terms of their Fourier transforms. Defining the Fourier component of a given function $f(x)$ by $f(\omega) = (1/2\pi)\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-\mathrm{i}\omega x} f(x)$, we have the following expressions for $A_{n,j}(\omega)$, $B_{r,r'}(\omega)$ and $\phi_{n,j}(\omega)$

$$A_{n,j}(\omega) = \coth(|\omega|/2)[\mathrm{e}^{-|n-j||\omega|/2} - \mathrm{e}^{-(n+j)|\omega|/2}] \tag{18}$$

$$B_{r,r'}(\omega) = \delta_{r,r'} - p(\omega)l_{r,r'} \tag{19}$$

$$\phi_{n,j/2}(\omega) = A_{n,j}(\omega)p(\omega) \tag{20}$$

where $p(\omega) = 1/2\cosh(\omega/2)$ and $l_{r,r'}$ is the incident matrix of the A_{N-1} Lie algebra.

The second step is to encode the temperature T via minimization of the free energy $F^{\mathrm{sym}} = E^{\mathrm{sym}} - TS$. By using a standard procedure [17–20] the energy E^{sym} and the entropy S can be written in terms of the densities of particles ($\sigma_n^r(\lambda)$) and holes ($\tilde{\sigma}_n^r(\lambda)$). After the minimization, $\delta F^{\mathrm{sym}} = 0$, we get the following thermodynamic Bethe ansatz (TBA) equations

$$\begin{aligned}\epsilon_n^r(\lambda) = {} & K_r(\lambda)(\delta_{n,1} + \delta_{n,k}) + T\sum_{r'}^{N-1}\varphi_{r,r'} * [\ln(1+\exp(\epsilon_n^{r'}/T))](\lambda) \\ & + T\sum_{r'}^{N-1}\tilde{\varphi}_{r,r'} * [\ln(1+\exp(\epsilon_{n+1}^{r'}/T)) + \ln(1+\exp(\epsilon_{n-1}^{r'}/T))](\lambda)\end{aligned} \tag{21}$$

where $\tilde{\sigma}_n^r(\lambda)/\sigma_n^r(\lambda) = \exp(\epsilon_n^r(\lambda)/T)$, $\epsilon_0^r(\lambda) \equiv 0$, and the Fourier component of the functions $K_r(\lambda)$, $\varphi_{r,r'}(\lambda)$ and $\tilde{\varphi}_{r,r'}(\lambda)$ are given by

$$K_r(\omega) = p(\omega)B_{r,1}^{-1}(\omega) \tag{22}$$

$$\varphi_{r,r'}(\omega) = \delta_{r,r'} - B_{r,r'}^{-1} \qquad \tilde{\varphi}_{r,r'}(\omega) = p(\omega)B_{r,r'}^{-1}. \tag{23}$$

Finally, the equilibrium free energy F^{sym} is given by

$$F^{\mathrm{sym}}/L = e_{\infty}^{\mathrm{sym}} - \frac{T}{4\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\sum_{r=1}^{N-1}K_r(\lambda)[\ln(1+\exp(\epsilon_1^r(\lambda)/T)) + \ln(1+\exp(\epsilon_k^r(\lambda)/T))] \tag{24}$$

One possible way to calculate the central charge of a conformally invariant system is by analysing the low-temperature behaviour of the respective free energy. The universal behaviour of the free energy is given by [21, 5]

$$F/L = e_\infty - \pi c T^2/6v_s. \tag{25}$$

It turns out that equations (21) and (24) allow us to make an exact calculation of such low-temperature behaviour. We first define the shift $\lambda \to \lambda - (N/2\pi)\ln(NT/2\pi)$, taking the derivative in λ of equation (21) and after some few standard manipulations [19] the $T \to 0$ limit can be expressed in terms of the dilogarithm functions $L(x)$ by

$$F^{\mathrm{sym}}/L = e_\infty^{\mathrm{sym}} - \frac{NT^2}{24}\left[(N-1)k - \sum_{r=1}^{N-1}\sum_{m=1}^{k-2} L\left(\frac{\sin[(k-m-1)\theta]\sin[m\theta]}{\sin[(m+r)\theta]\sin[(r+k-m-1)\theta]}\right)\right] \tag{26}$$

where

$$\theta = \frac{\pi}{N+k-1} \qquad \text{and} \qquad L(x) = -\frac{3}{\pi^2}\int_0^x \mathrm{d}t\left[\frac{\ln(t)}{1-t} + \frac{\ln(1-t)}{t}\right].$$

Using some identities for the sum of the dilogarithm function proved in [22], we finally have

$$F^{\mathrm{sym}}/L = e_\infty^{\mathrm{sym}} - \frac{T^2}{24}\frac{(N-1)((N+2)k-2)}{N+k-1}. \tag{27}$$

Comparing equations (25) and (27), we find that the central charge is $c = (N-1)((N+2)k-2)/(N+k-1)$. Remarkably enough, this conformal anomaly can be decomposed in terms of the central charges of two $SU(N)$ WZWN models with topological charge $\tilde{k} = 1$ ($c = N-1$) and $\tilde{k} = k-1$ ($c = (N^2-1)(k-1)/(N+k-1)$). This result generalizes similar decomposition mentioned by the authors [10] for the $SU(2)$ mixed Heisenberg model. In order to give extra support for this value of the central charge, we present some numerical results for the finite-size effects of the ground state energy in appendix B.

4. Discussions on possible generalizations

It is almost evident that all our discussion in section 2 can be generalized to an arbitrary representation of order k in the auxiliary space of states. The main technical difficulty is the explicit construction of the non-isomorphic vertex operator $R_{k,j}^{k'}(\mu)$. The solution of this problem has already been considered in [11] for arbitrary finite representations of $GL(2)$ Lie algebra. The vertex $R_{k,j}^{k'}(\mu)$ is expressed as a linear combination of certain projectors defined on the subspaces of the Klebsch–Gordon decomposition of $GL(2)_k \otimes GL(2)_{k'}$. The transfer matrix $T_{k,k'}(\mu)$ is then defined by

$$T_{k,k'}(\mu) = \mathrm{Tr}_{V^{(k)}}[R_{k,L}^{k}(\mu)R_{k',L-1}^{k}(\mu)\cdots R_{k,2}^{k}(\mu)R_{k',1}^{k}(\mu)]. \tag{28}$$

The eigenvectors and the eigenvalues of the associated one-dimensional Hamiltonian $H_{k,k'}$ can be determined by using the following strategy. We first define an auxiliary transfer matrix $T_{k,k'}^{\mathrm{aux}}(\mu)$ which commutes with $T_{k,k'}(\mu)$ as

$$\mathrm{T}_{k,k'}^{\mathrm{aux}}(\mu) = \mathrm{Tr}_{V^{(1)}}[R_{k,L}^{1}(\mu)R_{k',L-1}^{1}(\mu)\cdots R_{k,2}^{1}(\mu)R_{k',1}^{1}(\mu)]. \tag{29}$$

This definition has the advantage of reducing the auxiliary space to its fundamental representation, hence the standard QIS formalism can be applied. On the other hand, the eigenvalues of $T_{k,k'}(\mu)$ can be related to those of $T^{\mathrm{aux}}_{k,k'}(\mu)$ by certain commutators of $T_{k,k'}(\mu)$ and the usual QIS $B(\mu)$ operator [19]. The eigenvalues of the $H_{k,k'}$ are parametrized by the Bethe ansatz equation

$$\left(\frac{\lambda_j - \mathrm{i}k/2}{\lambda_j + \mathrm{i}k/2}\right)^{L/2} \left(\frac{\lambda_j - \mathrm{i}k'/2}{\lambda_j + \mathrm{i}k'/2}\right)^{L/2} = -\prod_{l=1}^{M} \frac{\lambda_j - \lambda_l - i}{\lambda_j - \lambda_l + i} \tag{30}$$

where $\mu_j = \mathrm{i}\lambda_j - 1/2$ and the eigenvalues $E_{k,k'}$ are determined by

$$E_{k,k'} = -\sum_{j=1}^{M} \frac{k}{(\lambda_j)^2 + (k/2)^2}. \tag{31}$$

Analogously to what we have discussed in section 2, a similar approach works for $T_{k',k}(\mu)$ and the conformally invariant quantum Hamiltonian can be defined through the product $T_{k,k'}(\mu)T_{k',k}(\mu)$. Considering the results of section 2 and those of [11] on the Yang–Baxter solutions for the $GL(N)$ group, it seems plausible that similar conclusions reached for $N = 2$ can be extended for an arbitrary value of N. Comparing the left-hand side of equations (7) and (30) in the case of $N = 2$, we observe that a certain factor $1/2$ has been replaced by $k/2$. This leads us to conjecture that a similar mechanism should work for a general mixed $GL(N)$ system. Taking this fact into account and writing equation (7) in a more convenient way, we conjecture that the form of a $GL(N)_k \otimes GL(N)_{k'}$ Bethe ansatz equation is

$$\left(\frac{\lambda_j^r - \mathrm{i}\delta_{r,1}k/2}{\lambda_j^r + \mathrm{i}\delta_{r,1}k/2}\right)^{L/2} \left(\frac{\lambda_j^r - \mathrm{i}\delta_{r,1}k'/2}{\lambda_j^r + \mathrm{i}\delta_{r,1}k'/2}\right)^{L/2} = -\prod_{r'=1}^{N-1}\prod_{l=1}^{M^{r'}} \frac{\lambda_j^r - \lambda_l^{r'} - \mathrm{i}C_{r,r'}/2}{\lambda_j^r - \lambda_l^{r'} + \mathrm{i}C_{r,r'}/2} \tag{32}$$

where $C_{r,r'}$ is the A_{N-1} Cartan matrix and the eigenvalues E^{sym} of the conformally invariant Hamiltonian are given by

$$E^{\mathrm{sym}} = -\sum_{j=1}^{M^1} \frac{k}{(\lambda_j^1)^2 + (k/2)^2} - \sum_{j=1}^{M^1} \frac{k'}{(\lambda_j^1)^2 + (k'/2)^2}. \tag{33}$$

It is not surprising that the structure of the Bethe ansatz equation is closely related to the A_{N-1} Lie algebra. In the case of homogeneous vertex models ($k = k'$), the authors of [23] have conjectured that the same structure will remain for all semi-simple A, D, E Lie algebras†. In fact in [24] this conjecture has been verified by an explicit computation in the case of a D_n Lie algebra. Based on these observations, let us assume that the same conjecture can be extented to the case of non-homogeneous ($k \otimes k'$) models. It is not difficult to verify that the associated TBA equations are similar to the system of equations (21). We just have to replace $l_{r,r'}$ and N in equations (21) by the incident matrix and the rank of the corresponding A, D, E Lie algebra. Interestingly enough, these equations can be cast in a rather useful form which will be helpful in the analysis of the low temperature. Defining the

† This fact is related to the idea that the classification of the solutions of the Yang–Baxter equations is somehow connected to the classification of the Lie algebras and their automorphisms.

function $Y_n^r(\lambda) = \exp(-\epsilon_n^r(\lambda)/T)$, making several manipulations in the Fourier transform of equation (21) and Fourier transforming back we find the following expression

$$Y_n^r(\lambda+\mathrm{i}/2)Y_n^r(\lambda-\mathrm{i}/2)\prod_{r'\in G}[1+Y_n^{r'}(\lambda)]^{-l_{r,r'}}\prod_{j\in A_\infty}[1+1/Y_j^r(\lambda)]^{l_{n,j}}$$
$$= \exp(2\pi\delta(\lambda)\delta_{r,1}(\delta_{n,k}+\delta_{n,k'})/T) \qquad (34)$$

where r' is an index characterizing the nodes of the Dynkin diagram of the $G \equiv A, D, E$ Lie algebra and j is a similar (unrestricted) index for the A_∞ Lie algebra.

Equation (34) defines a set of functional hierarchy relations for the functions $Y_n^r(\lambda)$. The possibility of constructing such functional relations from the TBA equations was first noted by Zamolodchikov [25] in the case of the diagonal system of scattering S-matrices. It also appears that certain functional hierarchies play the keystone in the computation of critical dimensions in integrable lattice models [26]. In our case they encode all the necessary information in order to obtain the low-temperature behaviour of the free energy. Procceding as in section 3, we can show that the $T \to 0$ limit of the free energy assumes the following form

$$F^{\text{sym}}/L = e_\infty^{\text{sym}} - \frac{T^2 h_G}{24\pi}\left[k'r_G - \sum_{r\in G}\sum_{j\in A_k} L(1/1+y_j^r(k)) - \sum_{r\in G}\sum_{j\in A_{k'-k}} L(1/1+y_j^r(k'-k))\right] \qquad (35)$$

where r_G and h_G are the rank and the dual Coxeter number of the Lie algebra G, and

$$e_\infty^{\text{sym}} = -\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{B_{1,1}^{-1}}{4\cosh(\omega/2)}[\phi_{k,k/2}(\omega) + \phi_{k',k'/2}(\omega) + 2\phi_{k',k/2}(\omega)].$$

The constants $y_j^r(m)$ satisfy the equation

$$y_j^r(m)^2 = \prod_{r'\in G}[1+y_j^{r'}(m)]^{l_{r,r'}}\prod_{i\in A_m}[1+1/y_i^r(m)]^{-l_{i,j}} \qquad (36)$$

where $l_{r,r'}$ ($l_{i,j}$) is the incident matrix of the Lie algebra G (A_m).

Remarkably enough, the sum of dilogarithm function [22, 27] appearing in equation (34) has been conjectured to have the expression

$$\sum_{r\in G}\sum_{j\in A_m} L(1/1+y_j^r(m)) = \frac{r_G m(m-1)}{h_G+m}. \qquad (37)$$

Although the proof of last identity was essentially given for the A and D Lie algebras, it can be verified directly by numerically solving equation (36) for several small values of m and h_G [27,6]. Using this dilogarithm sum and taking into account that now $v_s = 4\pi/h_G$, we finally obtain the following central charge

$$c = \frac{r_G k(h_G+1)}{h_G+k} + \frac{r_G(k'-k)(h_G+1)}{h_G+k'-k}$$

where we have already decomposed the result in terms of the central charge of two G invariant WZWN models with topological charges $\tilde{k} = k$ and $\tilde{k} = k' - k$, respectively. In particular for $k' = k$ we recover the known conjecture that rational isomorphic vertex models are in the class of universality of WZWN field theories [4–6].

5. Conclusion

In this paper we have discussed the critical behaviour of conformally invariant mixed $GL(N)$ spin chains. The central charge of an alternating model with 'spin' operators at order k (k') of representation acting on even (odd) sites can be decomposed in terms of WZWN field theories with topological charge k and $k'-k$ ($k'>k$). We have also considered the generalization of this result to other symmetric representations of the A, D, E Lie algebras.

Another possible extension of the results of this paper is to consider a collection of vertex operators at different representation distributed on a line of size L and with periodicity l. The associated rotational symmetric transfer matrix can be defined as

$$T^{\mathrm{sym}}_{k_1,\cdots,k_l}(\mu)=\prod_{i=1}^{l}\mathrm{Tr}_V^{(k_i)}[R^{k_i}_{k_1,L}(\mu)\cdots R^{k_i}_{k_l,L-l-1}(\mu)\cdots R^{k_i}_{k_1,l}(\mu)\cdots R^{k_i}_{k_l,1}(\mu)]. \tag{39}$$

Following our considerations of sections 3 and 4, the basic change in the TBA equations is that the right-hand side of equations (34) is replaced by $2\pi\delta(\lambda)\delta_{r,1}\sum_{i=1}^{l}\delta_{n,k_i}/T$. For instance, taking the following ordering $k_1<k_2<\cdots<k_l$, the central charge of the one-dimensional Hamiltonian associated with the system (39) will be

$$c=\sum_{i=0}^{l-1}c(k_{i+1}-k_i) \qquad k_0\equiv 0 \tag{40}$$

where $c(k)$ is the central charge of the A, D, E WZWN model with topological charge k.

Finally, it would be interesting to study the full operator content of these models and, in particular, to understand the decomposition of equation (40) in terms of the bosonic and parafermionic fields of the WZWN theories. We notice that for the sequence $k_{i+1}-k_i=1$ and $k_1=1$, the central charge is l multiplied by the rank of G and hopefully the operator content will be determined by lr_G coupled bosonic fields.

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Appendix A

Following the basic steps of the QLS method we propose a set of eigenstates $|\psi\rangle$ defined by [2, 3, 14]

$$|\psi\rangle\equiv\psi(\mu_1^1,\cdots,\mu_n^1)=F_{j_1\cdots j_{M^1}}B^{j_1}(\mu_1^1)\cdots B^{j_{M^1}}(\mu^1_{M^1})|0\rangle \tag{A.1}$$

where $|0\rangle$ is the pseudo vacuum (see equation (5)). The next step, motivated by the properties of $\tau_{1,k}(\mu)|0\rangle$, is to decompose the monodromy matrix as

$$\tau_{1,k}(\mu)=\begin{pmatrix} A(\mu) & B^i(\mu) \\ C^j(\mu) & D^{ij}(\mu)\end{pmatrix} \tag{A.2}$$

where $i(j)$ is a row(column) index, $i, j=1,\ldots,N-1$.

It follows from the identity $T_{1,k}(\mu) = \mathrm{Tr}_{V^{(1)}}[\tau_{1,k}(\mu)]$ that

$$T_{1,k}(\mu)|\psi\rangle = [A(\mu) + \sum_i D^{ii}(\mu)]|\psi\rangle = \Lambda(\mu, \{\mu_i^1\})|\psi\rangle. \tag{A.3}$$

In order to find the eigenstates $|\psi\rangle$ and the eigenvalues $\Lambda(\mu, \{\mu_i^1\})$ we also need the commutation relation between the $A(\mu)$, $B^i(\mu)$ and $D^{ii}(\mu)$ operators. These relations follows from the relation

$$\tilde{R}_1^1(\mu' - \mu)\tau_{1,k}(\mu') \otimes \tau_{1,k}(\mu) = \tau_{1,k}(\mu) \otimes \tau_{1,k}(\mu')\tilde{R}_1^1(\mu' - \mu) \tag{A.4}$$

where $\tilde{R}_1^1(\mu) = \mathcal{P}R_1^1(\mu)$. Using equations (2), (A.2) and (A.4) we have

$$\begin{aligned} A(\mu')B^i(\mu) &= \frac{\mu - \mu' + 1}{\mu - \mu'}B^i(\mu)A(\mu') + \frac{1}{\mu' - \mu}B^i(\mu')A(\mu) \\ D^{ii}(\mu')B^k(\mu) &= \frac{1}{\mu - \mu'}B^k(\mu')D^{ii}(\mu) + \frac{1}{\mu' - \mu}\sum B^k(\mu)D^{ii}(\mu')[R_1^1(\mu' - \mu)]^{lk,ni}_{[N-1]} \end{aligned} \tag{A.5}$$

where $[R_1^1(\mu' - \mu)]^{lk,ni}_{[N-1]}$ are the $GL(N-1)$ matrix elements of the matrix defined in equation (2).

From equations (A.1), (A.4) and (A.5) it follows that

$$\begin{aligned} A(\mu)|\psi\rangle &= \prod_{i=1}^{M^1} \frac{\mu_i^1 - \mu + 1}{\mu_i^1 - \mu} a(\mu)|\psi\rangle + \mathrm{UT} \\ D^{ii}(\mu)|\psi\rangle &= \left(\prod_{i=1}^{M^1} \frac{1}{\mu - \mu_i^1}\right) t^{(2)}{}^{i_1\cdots i_{M^1}}_{l_1\cdots l_{M^1}} F^{i_1\cdots i_{M^1}} B^{l_1}(\mu_1^1)\cdots B^{l_{M^1}}(\mu_{M^1}^1)d(\mu)|\psi\rangle + \mathrm{UT} \end{aligned} \tag{A.6}$$

where $a(\mu) = (\mu+1)^{L/2}(\mu + \frac{1}{2}(k+1)^{L/2}$, $d(\mu) = \mu^{L/2}(\mu - \frac{1}{2}(k-1))^{L/2}$ and $t^{(2)}{}^{i_1\cdots i_{M^1}}_{l_1\cdots l_{M^1}}$ are matrix elements of the following operator

$$t^{(2)}(\mu; \{\mu_j^1\}) = \mathrm{tr}[R^1_{1,M^1}(\mu - \mu^1_{M^1})_{[N-1]} \cdots R^1_{1,1}(\mu - \mu^1_1)_{[N-1]}] \tag{A.7}$$

and $F^{i_1\cdots i_{M^1}}$ are the eigenvectors' component of $t^{(2)}(\mu)$ with eigenvalues $\Lambda^{(2)}(\mu, \{\mu_i^1\})$.

UT stands for 'unwanted terms' which appear due to the interchange of the arguments μ and μ' in the relation (A.5). When these terms are null, $|\psi\rangle$ becomes an eigenstate of $T_{1,k}(\mu)$ and, as a consequence, we obtain a restriction to the rapidities μ (the Bethe ansatz equation). Finally, equation (A.7) is solved by introducing in each step $i = 2, \ldots, N-1$ a new matrix $t^{(i)}(\mu)$ acting on $M^{(i)}$ sites, analogously to that of equation (A.7). The final result for the eigenvalues $\Lambda(\mu)$ of $T_{1,k}(\mu)$ and $\Lambda^{(r)}(\mu, \{\mu_i^{r-1}\})$ of $t^{(r)}(\mu)$ are

$$\begin{aligned} \Lambda(\mu) &= a(\mu)\prod_{i=1}^{M^1}\frac{\mu_i^1 - \mu + 1}{\mu_i^1 - \mu} + d(\mu)\prod_{i=1}^{M^1}\frac{1}{\mu - \mu_i^1}\Lambda^{(2)}(\mu; \{\mu_j^{(1)}\}) \\ \Lambda^{(r)}(\mu; \{\mu_j^{r-1}\}) &= \prod_{i=1}^{M^{r-1}}(\mu - \mu_i^{r-1} + 1)\prod_{i=1}^{M^r}\frac{\mu_i^r - \mu + 1}{\mu_i^r - \mu} \\ &\quad + \prod_{i=1}^{M^{r-1}}(\mu - \mu_i^{r-1})\prod_{i=1}^{M^r}\frac{1}{\mu - \mu_i^r}\Lambda^{(r+1)}(\mu; \{\mu_j^r\}). \end{aligned} \tag{A.8}$$

Equation (7) is then obtained by imposing the zero residue condition in equation (A.8).

Appendix B

The critical behaviour of a conformally invariant theory can be determined by studying the consequences of the finite-size L effects for the eigenspectrum [28]. For example, the central charge is related to the ground-state energy $E^{\mathrm{sym}}(L)$ by [21, 5]

$$E^{\mathrm{sym}}(L)/L = e_{\infty}^{\mathrm{sym}} - \pi v_s c/L^2. \tag{B.1}$$

The central charge c can be numerically calculated by extrapolating the sequence

$$c(L) = -[E^{sym}(L)/L - e_{\infty}^{sym}]L^2/\pi v_s. \tag{B.2}$$

In table A1 we present our estimates for the sequence (B.2) in the case of $N = 2, 3$ (the $N = 2$ data have been already presented by us in [10]) and $k = 3$. We note that these numerical results are in accordance with the TBA analysis of section 3 (equation (27)). In our numerical analysis we have also observed that the case $k = 2$ is rather special. The string hypothes is $(\lambda_j^r = \xi_j^r \pm i/2)$ is almost exact for large enough L, presenting a very unusual small correction [8, 10]. In this case we can use the analytical method in [29] and conclude that the central charge is $c = 2(N - 1)$ (in agreement with equation (27)).

Table A1. The estimates of the central charge of equation (B.2) for $k = 3$ and $N = 2$ or $N = 3$.

L	$N = 2, k = 3$	L	$N = 3, k = 3$
8	2.839 364	12	5.614 431
16	2.602 543	24	5.336 131
24	2.556 301	36	5.277 662
32	2.538 530	48	5.254 475
40	2.529 503	60	5.242 437
48	2.524 142	72	5.235 167
Extrapolated	2.500(6)	Extrapolated	5.206(1)

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